

Secant Method



Vagelis Plevris and Abdulaziz Almutairi
Department of Civil and Environmental
Engineering, College of Engineering, Qatar
University, Doha, Qatar

Synonyms

Secant iterative method; Secant optimization method; Second-derivative-free Newton variant; Two-point Newton method; Two-point secant method for optimization

Definition

The secant method in single-variable optimization is a first-order iterative procedure that locates stationary points of a differentiable function without requiring explicit second-derivative information. Instead of using the exact curvature, as in Newton's method, it approximates the second derivative by a finite difference of successive first derivatives. Each iteration updates the estimate by effectively minimizing a locally linearized model of the derivative function, yielding super-linear convergence near a nondegenerate stationary point. As a second-derivative-free alternative to Newton's method, it is especially useful when evaluating or computing the second derivative is difficult or costly, while still offering faster

convergence than basic gradient-based methods. The method requires two initial guesses.

Historical Background and Importance

The secant method is a classical iterative technique with origins in early numerical analysis, closely related to the method of false position (*regula falsi*) (Ostrowski 1973) known since antiquity. In its modern form, it was formalized in the seventeenth to eighteenth centuries as an alternative to Newton's method for solving nonlinear equations (Burden and Faires 2015; Fletcher 2000). Unlike Newton's method for root finding, which uses the exact derivative, the secant method generates successive approximations to a root by using two prior iterates and replacing the exact derivative with a finite-difference approximation. The basic iteration is

$$x_{i+1} = x_i - f(x_i) \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}, \quad (1)$$

which provides an improved estimate of the root under suitable conditions. In contrast to false position, the secant method does not maintain a bracket, trading some global robustness for faster local progress.

This root-finding principle was later adapted to optimization. Instead of solving the equation $f(x) = 0$, the goal in optimization is to locate

stationary points where the first derivative vanishes, i.e., $f'(x) = 0$. Applying the secant idea to the derivative of the objective function naturally leads to the secant method for optimization. As a result, the optimization variant requires only function and gradient evaluations, while the second derivative is approximated through successive gradients. This makes the method derivative-free with respect to curvature information (second derivative), offering a practical alternative when second derivatives are difficult or costly to compute.

In the context of single-variable optimization, the secant method can be interpreted as an approximation to Newton's method in which the curvature term is replaced by a finite-difference estimate. Compared with Newton's method, it sacrifices quadratic convergence but achieves superlinear convergence near a nondegenerate stationary point (Nocedal and Wright 2006), striking a balance between efficiency and simplicity. Because of this property and its modest computational requirements, the secant method is recognized as one of the fundamental algorithms in numerical optimization. It continues to serve as a reference point for second-derivative-free optimization approaches and as a precursor to more advanced quasi-Newton strategies.

Mathematical Formulation

The secant method in optimization is based on modifying Newton's method to avoid the explicit computation of second derivatives. Recall that Newton's update for finding a stationary point requires the ratio of the first derivative to the second derivative of the objective function. The secant method approximates the second derivative by a finite difference of successive first derivatives, thereby reducing computational effort.

Consider a smooth scalar objective function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is continuously differentiable. The condition for a stationary point is given by $f'(x) = 0$. Applying the secant idea to the derivative function, the second derivative at iteration i is replaced by a finite-difference estimate:

$$f''(x_i) \approx \frac{f'(x_i) - f'(x_{i-1})}{x_i - x_{i-1}}. \quad (2)$$

Substituting this approximation into Newton's update leads to the secant method iteration for optimization:

$$\begin{aligned} x_{i+1} &= x_i - \frac{f'(x_i)}{\frac{f'(x_i) - f'(x_{i-1})}{x_i - x_{i-1}}} \\ &= x_i - f'(x_i) \frac{x_i - x_{i-1}}{f'(x_i) - f'(x_{i-1})}. \end{aligned} \quad (3)$$

This is the iterative scheme of the secant method for single-variable optimization. At each step, the new iterate is obtained by correcting the current point using the slope of the derivative function scaled by an approximate curvature. The process is repeated until convergence criteria are satisfied, typically when the first derivative becomes sufficiently small or when successive iterates differ by less than a prescribed tolerance.

Here $f'(x_i)$ is the local slope, while $[f'(x_i) - f'(x_{i-1})]/(x_i - x_{i-1})$ is the finite-difference approximation of the curvature (the numerically evaluated second derivative). In practice, the iteration stops when a convergence criterion is met or when a preset iteration limit is reached. Two common stopping criteria are when the step size becomes negligible or when the derivative approaches zero. Either condition indicates that the method is close to a stationary point.

Convergence Properties

The secant method in single-variable optimization is well known for its **superlinear convergence** under suitable conditions. Unlike Newton's method, which achieves quadratic convergence by relying on exact second derivatives, the secant method replaces curvature information with a finite-difference approximation of successive gradients. As a result, its convergence is typically slower than Newton's but significantly faster than first-order methods such as gradient descent.

The key requirements for superlinear convergence are:

- The function $f(x)$ must be continuously differentiable in a neighborhood of the solution.
- The stationary point x^* must satisfy $f'(x^*) = 0$ and $f''(x^*) \neq 0$, ensuring that the point is nondegenerate.
- The initial guesses x_0 and x_1 must be chosen sufficiently close to the solution so that the finite-difference approximation of the curvature is accurate.

When these conditions are satisfied, the secant method reduces the error at a rate proportional to approximately 1.618 (the golden ratio) per iteration (Dennis and Schnabel 1996; Ortega and Rheinboldt 2000). This is slower than quadratic convergence but still significantly more efficient than linear convergence. In practice, the method often requires only a modest number of iterations to achieve high accuracy.

Compared with Newton's method, the secant method has several distinctive convergence characteristics:

- **No need for explicit second derivatives:** Because the curvature is approximated, the method is more widely applicable when second derivatives are unavailable or expensive to compute.
- **Two starting points required:** Convergence relies on having two distinct initial guesses, which form the basis for the finite-difference curvature approximation. Poorly chosen starting points can lead to slow progress or divergence.
- **Smaller region of attraction:** Similar to Newton's method, the secant method is only locally convergent. Starting far from the true stationary point may lead to convergence to undesired points or failure to converge.
- **Sensitivity to flat regions:** If the difference $f'(x_i) - f'(x_{i-1})$ becomes very small, the correction step may be unstable or excessively large, leading to oscillations or divergence.

In practical implementations, convergence is usually checked by prescribing tolerances on both the change in iterates and the gradient magnitude. A common stopping set of rules is:

$$|x_i - x_{i-1}| < \varepsilon_x \text{ and/or } |f'(x_i)| < \varepsilon_g, \quad (4)$$

where ε_x and ε_g are small user-defined tolerances. These ensure that both the step size has stabilized and the stationarity condition is approximately satisfied.

Taken together, the secant method provides an attractive compromise between Newton's method and gradient-based techniques: it avoids the computational burden of second derivatives while still achieving superlinear convergence. Its effectiveness, however, remains highly dependent on good initialization and the smoothness of the objective function, which define its basin of attraction.

Variants

In practice, the basic secant method has been modified and extended in several ways to improve robustness and efficiency. One historical variant is the method of false position (*regula falsi*), which also employs a secant line but enforces bracketing of the solution, generally resulting in slower but more stable convergence. More advanced approaches such as inverse quadratic interpolation replace the single secant approximation with a quadratic interpolant through three points, often accelerating convergence, and are frequently combined with bracketing strategies in hybrid algorithms like Brent's method (Dennis and Schnabel 1996). Multi-point secant methods generalize the idea further by using several past iterates to approximate curvature, trading simplicity for improved stability. In optimization, hybrid Secant–Newton approaches are sometimes employed, where the secant method is used initially to avoid costly second-derivative computations and the iteration switches to Newton's method once the iterates are close to a stationary point. Finally, the secant

principle also underlies quasi-Newton methods in multivariable optimization, where approximate Hessians are updated to satisfy a secant condition, making the single-variable secant method the simplest representative of this broader family (Fletcher 2000).

Applications

The secant method for single-variable optimization is primarily used in situations where first derivatives of the objective function can be computed or estimated, but second derivatives are either unavailable, expensive, or numerically unstable. Typical applications include small-scale optimization problems in engineering and the sciences where efficiency and simplicity are important. In teaching and benchmarking, the method is often presented alongside Newton's method and gradient descent to illustrate the trade-offs between computational cost and convergence speed.

Beyond its direct use, the secant method is conceptually important as the foundation of quasi-Newton algorithms, which extend the secant idea to multivariable optimization by building approximate Hessian matrices. These methods, such as the historically significant Davidon–Fletcher–Powell (DFP) algorithm and the widely used Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm, iteratively construct an approximation of the Hessian matrix using only first-derivative (gradient) information. This makes them highly effective for complex, high-dimensional problems, and they are widely applied in engineering design optimization, data fitting, and machine learning, demonstrating the broader influence of the secant principle (Bottou et al. 2018; Rao 2009). In practice, the single-variable secant method is most effective when used as an intermediate strategy between simple gradient-based methods and full second-order approaches, offering a balance between low per-iteration cost and superlinear convergence.

Advantages and Limitations

The secant method offers several practical advantages for single-variable optimization. Its most notable benefit is that it requires only first-derivative information, avoiding the explicit evaluation of the second derivative $f''(x)$. This makes the method appealing in situations where curvature information is difficult, costly, or unreliable to compute. Because the curvature is approximated using a finite difference of successive gradients, each iteration is inexpensive and easy to implement. When the initial guesses are sufficiently close to a nondegenerate stationary point, the method achieves superlinear convergence, often reducing the error at a rate of approximately 1.618 per iteration. This improvement over purely first-order approaches gives the secant method an attractive balance between simplicity and efficiency (Kelley 1999).

Several limitations accompany these strengths. The method requires two distinct initial points, and its overall performance depends on selecting these points within a suitable region of attraction. Poorly chosen initial guesses can lead to slow convergence, oscillations, or divergence, especially if the finite-difference approximation of curvature becomes inaccurate. Sensitivity arises in particular when the difference $f'(x_i) - f'(x_{i-1})$ becomes very small (Dennis and Schnabel 1996). In such cases, the secant update may produce excessively large or unstable steps. As with many local optimization algorithms, the method provides no guarantee of global convergence, and safeguards such as damping or line-search strategies are commonly introduced to stabilize the iteration when curvature estimates are unreliable (Ortega and Rheinboldt 2000).

Because it avoids reliance on exact second-derivative information, the secant method is especially useful when evaluating $f''(x)$ is impractical or would incur significant computational cost. However, this benefit is accompanied by a typically smaller region of attraction and the need for careful initialization. In well-behaved problems with accessible gradients, the method provides a

practical compromise between low per-iteration cost and accelerated local convergence, making it a valuable tool in derivative-based optimization.

Algorithm and Implementation

Goal. Minimize a continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$.

Step-by-step guide:

Step 1. Initialization

Choose two distinct initial guesses x_0 and x_1 for the minimizer. Set the iteration counter $i = 1$ and specify a maximum number of iterations. Define convergence tolerances for the step size and the gradient magnitude.

Step 2. Compute derivative

At the current point x_i , evaluate the first derivative $f'(x_i)$.

Step 3. Update

Form the Secant iterate using the update formula of Eq. (3).

Step 4. Convergence check

After computing the new iterate, test for convergence. Stop if the step size $|x_i - x_{i-1}|$ is below the prescribed tolerance, if the gradient magnitude $|f'(x_i)|$ is sufficiently small, or if the maximum number of iterations has been reached. Other termination criteria may also be used.

Step 5. Result

If a stopping condition is satisfied, accept the current point x_i as an approximation of the solution. To classify the point, check the second derivative, if available: positive indicates a local minimum, negative a local maximum, and near zero an inflection point. Otherwise, set $i \leftarrow i + 1$ and return to Step 2.

Numerical Example

For consistency and comparison, the secant method is applied to the same test function considered in the Newton's method chapter:

$$f(x) = x^4 - 3x^3 + 2, \quad (5)$$

with the goal of locating a stationary point in the interval $x \in [-1, 3]$. Two initial guesses are used, $x_0 = 3$ and $x_1 = 2.5$.

Solution

The first derivative of f is computed:

$$f'(x) = 4x^3 - 9x^2. \quad (6)$$

Using the expression for the first derivative, the secant method is applied using Eq. (3), starting from the initial guesses $x_0 = 3$ and $x_1 = 2.5$. The iteration results are summarized in Table 1.

The procedure is terminated at iteration 6, since at this point the derivative magnitude (or the step size) is below a chosen tolerance. The computed minimizer is

$$x^* = 2.25, f(x^*) \approx -6.54297. \quad (7)$$

Compared with Newton's method, the secant method requires one or two additional iterations to reach comparable accuracy. This difference reflects the trade-off between avoiding second derivatives and achieving a slower, superlinear convergence rate. Nevertheless, the method successfully identifies the minimizer with only modest computational effort and demonstrates its value as a practical second-derivative-free alternative.

To illustrate sensitivity to initialization, the experiment is repeated with the initial points $x_0 = 1$ and $x_1 = 0.5$. The results are presented in Table 2. In this case, the method converges toward $x \approx 0$, which is an inflection point of

Secant Method, Table 1 Iteration results for the secant method applied to $f(x)$ with $x_0 = 3$ and $x_1 = 2.5$

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	Step size, $ x_i - x_{i-1} $
0	3	2	27	–
1	2.5	–5.8125	6.25	0.5
2	2.349398	–6.43695	2.194567	0.150602
3	2.2679	–6.53969	0.36827	0.081497
4	2.251466	–6.54295	0.029733	0.016434
5	2.250023	–6.54297	0.000466	0.001443
6	2.25	–6.54297	6.07×10^{-7}	2.3×10^{-5}

the function rather than a local minimum. This behavior parallels the Newton iteration on the same function and highlights the importance of suitable starting values for both methods.

Figure 1 illustrates this behavior by showing the secant method paths from the two different starting point pairs.

The black curve represents $f(x)$. The blue path corresponds to the pair $[x_0, x_1] = [3, 2.5]$ and

shows the secant iterates moving monotonically toward the true minimizer near $x \approx 2.25$; the final iterate is marked with an asterisk. The orange path corresponds to $[x_0, x_1] = [1, 0.5]$. In this case, the sequence drifts left and converges to $x = 0$, which is a stationary inflection point of f rather than a local minimum. The plot makes the basins of attraction visible: starting pairs to the right of roughly $x \approx 1.5$ are drawn toward the minimizer at 2.25, while pairs on the left tend to the stationary point at 0. Compared with Newton’s paths from the same starts, the secant trajectories are qualitatively similar but require more iterations to reach comparable accuracy, reflecting superlinear—rather than quadratic—convergence.

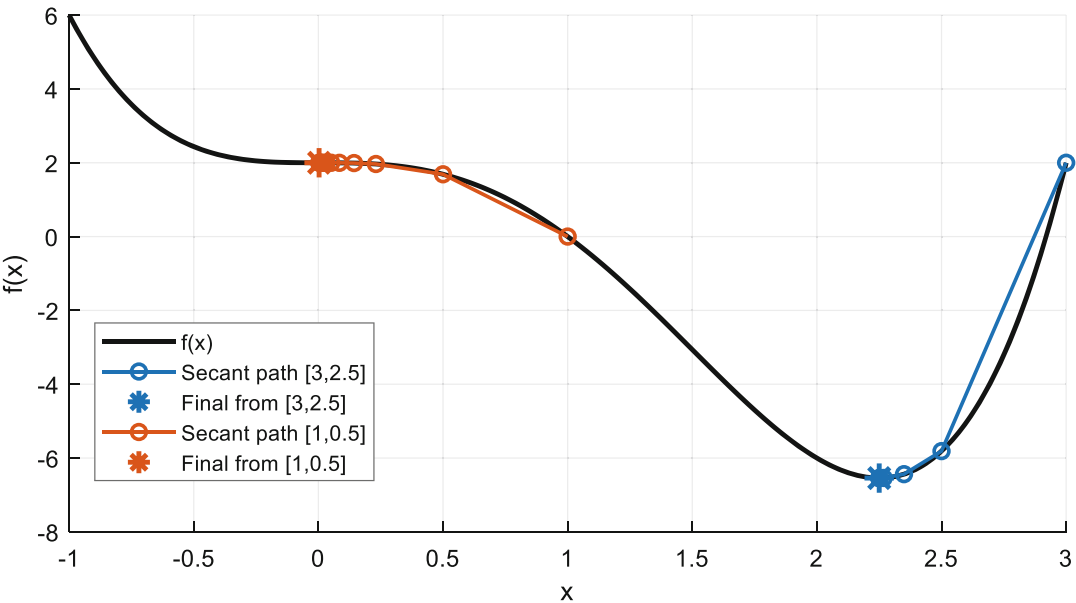
Secant Method, Table 2 Iteration results for the secant method applied to $f(x)$ with $x_0 = 1$ and $x_1 = 0.5$

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	Step size, $ x_i - x_{i-1} $
0	1	0	−5	–
1	0.5	1.6875	−1.75	0.5
2	0.230769	1.965968	−0.43013	0.269231
3	0.14303	1.99164	−0.17241	0.08774
4	0.084332	1.998251	−0.06161	0.058698
5	0.051696	1.999593	−0.0235	0.032636
6	0.031571	1.999907	−0.00884	0.020125
7	0.019425	1.999978	−0.00337	0.012146
8	0.01196	1.999995	−0.00128	0.007465
9	0.007378	1.999999	−0.00049	0.004582
10	0.004554	2	−0.00019	0.002824
11	0.002812	2	-7.1×10^{-5}	0.001742
12	0.001737	2	-2.7×10^{-5}	0.001075

Cross-References

- [Gradient Descent](#)
- [Newton’s Method](#)

Competing Interest Declaration The author(s) has no competing interests to declare that are relevant to the content of this manuscript.



Secant Method, Fig. 1 Secant method iteration paths from two starting-point pairs

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